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On the Iterates of Some Bernstein-Type Operators*

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In this paper, we establish two basic functional-type identities between the iterates of the Bleimann–Butzer–Hahn operator and those of the Bernstein operator, on the one hand, and the iterates of the (modified) Meyer–König and Zeller operator and those of the Baskakov operator, on the other. These identities allow us to transfer the properties of these operators from one to another. Attention is focused on the limit behavior of the iterates and the linear combinations of iterates of Fejér–Korovkin type. © 1997 Academic Press

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1. INTRODUCTION

The linear operator L_n defined by

$$L_n(f, x) := \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k}\frac{x^k}{(1+x)^n}, \qquad x \geq 0,\ n = 1, 2, \ldots,$$

where $f \in \mathbb{R}^{[0,\infty)}$ (for convenience, we also set $L_0(f, x) := f(0)$) was introduced by Bleimann, Butzer, and Hahn [1] to approximate continuous functions on the positive semi-axis and has been studied by several authors (see, for instance, [3, 9, 10, 14]).

In [1], the authors pointed out some formal similarities and differences between L_n and other operators, namely, the classical Bernstein operator given by

$$B_n(h, x) := \sum_{k=0}^{n} h\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}, \qquad x \in [0, 1],\ n = 1, 2, \ldots,$$

where $h \in \mathbb{R}^{[0,1]}$; the Baskakov operator H_n defined by

$$H_n(f, x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}, \qquad x \geq 0,\ n = 1, 2, \ldots,$$

where $f \in C[0,\infty)$ satisfies $f(x) = O(x^a)$ $(x \to \infty)$, for some $a > 0$; and the Meyer-König and Zeller operator M_n (in the modified version of Cheney and Sharma [2]) defined by

$$M_n(g, x) := (1-x)^{n+1}\sum_{k=0}^{\infty} g\left(\frac{k}{n+k}\right)\binom{n+k}{k}x^k,$$
$$x \in [0, 1),\ n = 1, 2, \ldots,$$

where $g \in C[0, 1)$ satisfies $g(x) = O((1-x)^{-a})$, $(x \to 1)$, for some $a > 0$.

In a further remark (after [1, Lemma 1]), the authors observed that "there exists essentially the same connection between the B_n and the L_n as between the H_n and the M_n; it is basically given by the rational transformation $r(u) := u/(1+u)$, $u \in [0,\infty)$, and its inverse $r^{-1}(v) := v/(1-v)$, $v \in [0, 1)$. Since this transformation is rational, one can hardly expect to carry over the well-known results of the operator B_n (or H_n) to the transformed operator L_n (or M_n)." This last assertion is true with regard to convergence results (with rates), which is the main topic considered in [1]. However, if one is interested in various kinds of iterates of these operators and other topics, the preceding ideas become very fruitful.

In the next section, we show that the "connection" suggested in [1] can be formulated in a precise way by means of the following two identities (Theorems 1 and 2),

$$L_n^k = T_* \circ B_{n+1}^k \circ S_*, \tag{1}$$

$$M_n^k = T \circ H_n^k \circ S, \tag{2}$$

where A^k denotes the kth iterate of the operator A (i.e., $A^0 = I$ is the identity operator on the corresponding function space and, for $k \geq 1$, $A^k = A \circ A^{k-1}$), and S, T, S_*, T_* are suitable positive linear operators which will be defined below.

Thanks to identity (1), different properties of B_n can be transferred to L_n with little extra effort. Thus, the limiting behavior of the iterates of L_n is immediately derived from (1) and the well-known results by Kelisky and Rivlin [8] and Karlin and Ziegler [7] for the iterates of B_n (see Theorem 5 and Remark 1 below). On the other hand, the behavior of the iterates of Fejér–Korovkin type

$$L_{n,r} := I - (I - L_n)^r = \sum_{j=1}^{r} (-1)^{j-1} \binom{r}{j} L_n^j,$$

is also derived from (1) and the corresponding result for B_n shown by Felbecker [4] (see Theorem 4 in Section 3). Identity (2) is used to carry over analogous properties of the operator H_n to the operator M_n (Theorem 3, Remark 2). Finally, formulae (1) and (2) may have other applications, some of which are outlined in the last section.

2. THE BASIC IDENTITIES

We introduce the following auxiliary operators. Firstly, let $T: \mathbb{R}^{[0,1)} \to \mathbb{R}^{[0,\infty)}$ and $T_*: \mathbb{R}^{[0,1]} \to \mathbb{R}^{[0,\infty)}$ be the positive linear operators defined by

$$T(g, x) := (1 + x) g(x/1 + x), \qquad x \in [0, \infty),$$

$$T_* h := T h_*,$$

where h_* is the restriction of h to $[0,1)$. Secondly, let $S: \mathbb{R}^{[0,\infty)} \to \mathbb{R}^{[0,1)}$ and $S_*: \mathbb{R}^{[0,\infty)} \to \mathbb{R}^{[0,1]}$ be defined by

$$S(f, x) := (1 - x) f(x/1 - x), \qquad x \in [0, 1),$$

$$S_*(f, x) := \begin{cases} S(f, x) & \text{if } x \in [0, 1) \\ 0 & \text{if } x = 1. \end{cases}$$

On the other hand, we shall use the notations

$$M := \left\{g \in C[0,1) \colon g(x) = O\left((1-x)^{-a}\right)\ (x \to 1), \text{ for some } a > 0\right\},$$
$$H := \{f \in C[0,\infty) \colon f(x) = O(x^a)\ (x \to \infty), \text{ for some } a > 0\},$$
$$C^*[0,\infty) := \{f \in C[0,\infty) \colon f(x) = o(x), (x \to \infty)\}.$$

The following lemma collects some properties of the preceding operators to be used throughout the paper. All the assertions can be checked by elementary calculations.

LEMMA 1. *We have*

(a) $T \circ S = T_* \circ S_*$ *is the identity operator on* $\mathbb{R}^{[0,\infty)}$.

(b) $S \circ T$ *is the identity operator on* $\mathbb{R}^{[0,1)}$.

(c) $(S_* \circ T_*)h = h$, *for every* $h \in \mathbb{R}^{[0,1]}$ *such that* $h(1) = 0$.

(d) S, T, *and* T_* *preserve continuity and convergence on compact subsets*.

(e) *If* $f \in C^*[0,\infty)$ *then* $S_* f \in C[0,1]$.

(f) *Let* $r = 0, 1, \dots$. *If* $g \in C^r[0,1)$ *satisfies* $g^{(r)} \in M$, *then* $Tg \in C^r[0,\infty)$ *and* $(Tg)^{(r)} \in H$.

(g) *Let* $r = 0, 1, \dots$. *If* $f \in C^r[0,\infty)$ *satisfies* $f^{(r)} \in H$, *then* $Sf \in C^r[0,1)$ *and* $(Sf)^{(r)} \in M$.

The relation between the iterates of L_n and B_{n+1} is given in the following theorem.

THEOREM 1. *For* $n, k = 0, 1, 2 \dots$, *we have*

$$L_n^k = T_* \circ B_{n+1}^k \circ S_*.$$

Proof. If $k = 0$, we just have Lemma 1(a). If $k = 1$, the result follows from the equality

$$\sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k}\frac{x^k}{(1+x)^n}$$
$$= (1+x)\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right)f\left(\frac{k/n+1}{1-(k/n+1)}\right)$$
$$\times\binom{n+1}{k}\left(\frac{x}{1+x}\right)^k\left(\frac{1}{1+x}\right)^{n+1-k},$$

where $f \in \mathbb{R}^{[0,\infty)}$ and $x \geq 0$. Finally, if $k \geq 2$, the conclusion follows by induction on k, taking into account Lemma 1(c). ∎

Before stating the identities concerning the operators M_n and H_n, we give the following two lemmata. The first one is a well-known elementary result (cf. [6]).

LEMMA 2. *Let X be a random variable having the negative binomial distribution with parameters n and $q \in (0,1)$, i.e.,*

$$P(X = j) = \binom{n+j-1}{j}(1-q)^n q^j, \qquad j = 0,1,2,\ldots.$$

Then, for $k = 1,2,\ldots,$

$$E[X(X-1)\cdots(X-k+1)] = n(n+1)\cdots(n+k-1)(q/1-q)^k.$$

The second lemma guarantees that M_n and H_n can be iterated on M and H, respectively.

LEMMA 3. *For $n = 1,2,\ldots,$ we have:*

(a) *If $g \in \mathsf{M}$ then $M_n g \in \mathsf{M}$.*

(b) *If $f \in \mathsf{H}$ then $H_n f \in \mathsf{H}$.*

Proof. Both parts (a) and (b) easily follow from Lemma 2 and the probabilistic representations

$$H_n(f,x) = Ef\left(\frac{V_n(x/1+x)}{n}\right),$$

$$M_n(g,x) = Eg\left(\frac{V_{n+1}(x)}{n+V_{n+1}(x)}\right),$$

where E denotes mathematical expectation and $V_n(u)$ is a random variable having the negative binomial distribution with parameters n, u. ∎

THEOREM 2. *For $n = 1,2,\ldots$ and $k = 0,1,2,\ldots,$ we have*

$$M_n^k g = \left(S \circ H_n^k \circ T\right)g, \qquad g \in \mathsf{M},$$

and

$$H_n^k f = \left(T \circ M_n^k \circ S\right)f, \qquad f \in \mathsf{H}.$$

Proof. The proof follows along the lines of that in Theorem 1. Details are omitted. ∎

3. ITERATES OF FEJER–KOROVKIN TYPE

In order to increase the rate of convergence according to the smoothness of the functions, several authors (cf. [4, 12, 13]) have considered the operator $B_{n,r}$ defined by

$$B_{n,r} := I - (I - B_n)^r = \sum_{j=1}^{r} (-1)^{j-1} \binom{r}{j} B_n^j, \qquad n, r = 1, 2, \dots .$$

Obviously, $B_{n,1} = B_n$. For $r > 1$, the operator $B_{n,r}$ is not positive, but it satisfies the following property shown by Felbecker [4].

THEOREM A. *Let $r = 1, 2, \dots$ and $h \in C^{2r}[0,1]$. Then*

$$\lim_{n\to\infty} n^r \left[B_{n,r}(h, x) - h(x) \right] = (-1)^{r-1} B_*^r(h, x)$$

uniformly on $[0,1]$, where B_* is the differential operator defined by

$$B_*(h, x) := \frac{x(1-x)}{2} h''(x), \qquad x \in [0,1], h \in C^2[0,1]. \tag{3}$$

This result has been extended by Gawronsky and Stadtmüller [5] to a large class of discrete operators. The following theorem is the specialization of [5, Theorem 1] to the case of the Baskakov operator.

THEOREM B. *Let $r = 1, 2, \dots$ and let $f \in C^{2r}[0,\infty)$ such that $f^{(2r)} \in \mathsf{H}$. Then*

$$\lim_{n\to\infty} n^r \left[H_{n,r}(f, x) - f(x) \right] = (-1)^{r-1} H^r(f, x)$$

uniformly on compact subsets of $[0,\infty)$, where $H_{n,r} := I - (I - H_n)^r$ and H is the differential operator defined by

$$H(f, x) := \frac{x(1+x)}{2} f''(x), \qquad x \ge 0, f \in C^2[0,\infty). \tag{4}$$

The results of Gawronsky and Stadtmüller do not apply to L_n nor M_n. In this section, the analogous results for L_n and M_n are derived from Theorems A and B, respectively, by using the identities shown in the preceding section. To do this, we need the following auxiliary result.

LEMMA 4. (a) *Let L be the differential operator given by*

$$L(f,x) := \frac{x(1+x)^2}{2} f''(x), \qquad x \geq 0, f \in C^2[0,\infty). \tag{5}$$

Then, we have

$$L^r = T \circ B^r \circ S, \qquad r = 1, 2, \ldots,$$

where B is the differential operator on $C^2[0,1)$ defined in the same way as B_* in (3), *i.e.*,

$$B(g,x) := \frac{x(1-x)}{2} g''(x), \qquad x \in [0,1), g \in C^2[0,1).$$

(b) *Let M be the differential operator given by*

$$M(g,x) := \frac{x(1-x)^2}{2} g''(x), \qquad x \in [0,1), g \in C^2[0,1). \tag{6}$$

We have

$$M^r = S \circ H^r \circ T, \qquad r = 1, 2, \ldots,$$

where H is defined in (4).

Proof. Parts (a) and (b) have similar proofs. Actually, the case $r = 1$ can be easily checked and, for $r \geq 2$, the conclusion follows by induction, taking into account Lemma 1(a), (b). ∎

Using Theorem 2, Lemma 1, and Lemma 4(b), the following result is obtained as an immediate consequence of Theorem B.

THEOREM 3. *Let $r = 1, 2, \ldots$ and let $g \in C^{2r}[0,1)$ such that $g^{(2r)} \in \mathsf{M}$. Then*

$$\lim_{n\to\infty} n^r [M_{n,r}(g,x) - g(x)] = (-1)^{r-1} M^r(g,x)$$

uniformly on compact subsets of $[0,1)$, *where*

$$M_{n,r} := I - (I - M_n)^r = S \circ H_{n,r} \circ T$$

and M is the differential operator defined in (6).

The corresponding result for the operator L_n is stated in Theorem 4. Observe that it contains, as a particular case, the Voronovskaja-type result for L_n obtained by Totik [14].

THEOREM 4. *Let $r = 1, 2, \ldots$ and let $f \in \mathsf{H} \cap C^{2r}[0,\infty)$. Then*

$$\lim_{n\to\infty} n^r\left[L_{n,r}(f,x) - f(x)\right] = (-1)^{r-1} L^r(f,x)$$

uniformly on compact subsets of $[0,\infty)$, *where*

$$L_{n,r} := I - (I - L_n)^r = T_* \circ B_{n+1,r} \circ S_* \tag{7}$$

and L is the differential operator given in (5).

The proof of Theorem 4, based on Theorem 1, Lemma 1, and Theorem A, is not as simple as that of Theorem 3. In fact, if $f \in C^{2r}[0,\infty)$, then $S_* f \in C^{2r}[0,1)$, but it cannot be guaranteed that $S_* f \in C^{2r}[0,1]$. This is the reason why we shall need the following auxiliary result.

LEMMA 5. *Let $x \in [0,1]$ and $n = 1, 2, \ldots$. Define, inductively on j,*

$$p_{n,k}^{(1)}(x) := \binom{n}{k} x^k (1-x)^{n-k}, \qquad k = 0, \ldots, n,$$

$$p_{n,k}^{(j)}(x) := \sum_{i=0}^{n} p_{n,k}^{(1)}(i/n)\, p_{n,i}^{(j-1)}(x), \qquad k = 0, \ldots, n.$$

For $j = 1, 2, \ldots,$ we have

(a) *Let $h \in \mathbb{R}^{[0,1]}$. Then*

$$B_n^j(h,x) = \sum_{k=0}^{n} h(k/n)\, p_{n,k}^{(j)}(x).$$

In particular (*take $h \equiv 1$*), *the quantities $p_{n,k}^{(j)}(x)$, $k = 0, \ldots, n$ add up to* 1.

(b) *For $p \geq 1$,*

$$\sum_{k=0}^{n} \left|\frac{k}{n} - x\right|^p p_{n,k}^{(j)}(x) \leq C_{j,p}\, n^{-p/2}, \qquad n = 1, 2 \ldots,$$

where $C_{j,p}$ is some positive constant only depending upon j and p.

Proof of Lemma 5. Assertion (a) can be easily checked by induction on j. To prove (b), we also use induction. For $j = 1$, the result is well known (cf., for instance, [11, p. 15]). On the other hand, using the inequality

$$|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p), \qquad a, b \in \mathbb{R},$$

we have, for $j > 1$,

$$\sum_{k=0}^{n} \left| \frac{k}{n} - x \right|^{p} p_{n,k}^{(j)}(x) = \sum_{k=0}^{n} \sum_{i=0}^{n} \left| \frac{k}{n} - x \right|^{p} p_{n,k}^{(1)}(i/n) p_{n,i}^{(j-1)}(x)$$

$$\leq 2^{p-1} \left[\sum_{i=0}^{n} \left(\sum_{k=0}^{n} \left| \frac{k}{n} - \frac{i}{n} \right|^{p} p_{n,k}^{(1)}(i/n) \right) p_{n,i}^{(j-1)}(x) \right.$$

$$\left. + \sum_{i=0}^{n} \left| \frac{i}{n} - x \right|^{p} p_{n,i}^{(j-1)}(x) \right].$$

Thus, the conclusion follows from (a), the case $j = 1$, and the induction hypothesis. ∎

Now, we are in a position to show Theorem 4.

Proof of Theorem 4. Let $a > 0$ and fix z such that $a^* := a/1 + a < z < 1$. Choose $h \in C^{2r}[0,1]$ such that

$$h(u) = S_*(f, u), \qquad \text{for all } u \in [0, z]. \tag{8}$$

By (7) and (8), we obviously have, for $x \in [0, a]$,

$$L_{n,r}(f, x) - f(x) = Q_n(x) + R_n(x),$$

where

$$Q_n(x) := (1 + x)[B_{n+1,r}(h, x/1 + x) - h(x/1 + x)]$$

and

$$R_n(x) := (1 + x) B_{n+1,r}(S_* f - h, x/1 + x).$$

From Theorem A, Lemma 4(a), and (8), we deduce that

$$\lim_{n \to \infty} n^r Q_n(x) = (1 + x)(-1)^{r-1} B_*^r(h, x/1 + x)$$

$$= (-1)^{r-1} (T \circ B^r \circ S)(f, x)$$

$$= (-1)^{r-1} L^r(f, x),$$

uniformly on $[0, a]$. Therefore, the proof will be complete as soon as we show that, for $j = 1, \ldots, r$,

$$\sup_{x \in [0,a]} B_{n+1}^{j}(|S_* f - h|, x/1 + x) \leq C_j^* n^{-(r+1)}, \qquad n = 1, 2, \ldots, \tag{9}$$

where C_j^* is some positive constant.

Since $f \in H$, it is not hard to see that

$$\sup_{k/(n+1)>z} |S_*(f, k/n+1)| \le Cn^b, \qquad n = 1, 2, \dots, \tag{10}$$

for some constants $C > 0$ and $b \ge 0$. Taking into account (8), and using successively (10), Chebyshev's inequality, and Lemma 5, we obtain, for $x \in [0, a]$.

$$\begin{aligned}
&B_{n+1}^{j}(|S_* f - h|, x/1+x) \\
&\quad = \sum_{k/(n+1)>z} |S_*(f, k/n+1) - h(k/n+1)| p_{n+1,k}^{(j)}(x/1+x) \\
&\quad \le (Cn^b + \|h\|) \sum_{|k/(n+1) - x/(1+x)| > (z-a^*)} p_{n+1,k}^{(j)}(x/1+x) \\
&\quad \le C' n^b (z - a^*)^{-2(b+r+1)} \sum_{k=0}^{n+1} \left| \frac{k}{n+1} - \frac{x}{1+x} \right|^{2(b+r+1)} \\
&\qquad \times p_{n+1,k}^{(j)}(x/1+x) \\
&\quad \le C'' n^b C_j (n+1)^{-(b+r+1)} \le C_j^* n^{-(r+1)},
\end{aligned}$$

where $\|\cdot\|$ denotes the sup-norm in $C[0,1]$. This shows claim (9) and, therefore, the proof of Theorem 4 is complete. ∎

4. LIMIT BEHAVIOR OF ITERATES

The main purpose of this section is to deduce the limit behavior of $L_n^{k(n)}$, as $n \to \infty$ and $k(n) \to \infty$, from the limit behavior of $B_n^{k(n)}$.

The following theorem summarizes the main results concerning the limit behavior of the iterates of the Bernstein operator. Parts (a), (b), and formula (11) were obtained by Kelisky and Rivlin [8], while the first assertion in part (c) was shown by Karlin and Ziegler [7].

THEOREM C. *Let $h \in C[0,1]$. We have*:

(a) *If $k(n)/n \to 0$ then $\|B_n^{k(n)} h - h\| \to 0$.*

(b) *If $k(n)/n \to \infty$ then $\|B_n^{k(n)} h - B_1 h\| \to 0$.*

(c) *If $k(n)/n \to t \in (0, \infty)$ then $\|B_n^{k(n)} h - B(t) h\| \to 0$, where $\{B(t): t \ge 0\}$ is the C_0-semigroup of operators acting on $C[0,1]$ whose infinitesimal generator is the differential operator B_* defined in* (3). *In particular, if $h_r(u) := u^r$ $(r = 1, 2, \dots)$, then*

$$B(t)(h_r, x) = \sum_{i=1}^{r} b_i x^i, \qquad x \in [0,1],$$

where

$$b_i := \frac{i}{r}\binom{r}{i}^2 \sum_{j=i}^{r} \frac{(-1)^{i+j}\binom{r-i}{j-i}^2}{\binom{2j-2}{j-i}\binom{j+r-1}{r-j}} e^{-j(j-1)t/2}. \tag{11}$$

Combining Theorem C with Theorem 1 and Lemma 1, we immediately obtain the following result.

THEOREM 5. *Let $f \in C^*[0,\infty)$. We have*:

(a) *If $k(n)/n \to 0$ then $L_n^{k(n)}(f,x) \to f(x)$, uniformly on compact subsets*.

(b) *If $k(n)/n \to \infty$ then $L_n^{k(n)}(f,x) \to f(0)$, uniformly on compact subsets*.

(c) *If $k(n)/n \to t \in (0,\infty)$ then $L_n^{k(n)}(f,x) \to L(t)(f,x)$, uniformly on compact subsets, where*

$$L(t) := T_* \circ B(t) \circ S_*. \tag{12}$$

In particular, if $f_r(u) := (1+u)^{1-r}$ $(r = 1,2,\dots)$, then

$$L(t)(f_r,x) = \sum_{i=1}^{r} b_i (1+x)^{1-i}, \qquad x \geq 0,$$

where b_i is defined in (11).

Remark 1. Karlin and Ziegler [7] have shown that

$$B(t)(h,x) = B_1(h,x) + \int_0^1 [h(y) - B_1(h,y)]\, p(t;x,y)\, dy,$$

where $p(t;x,y)$ is the transition probability density of a diffusion process on $[0,1]$ with absorbing barriers whose backward equation is

$$\frac{\partial p}{\partial t} = \frac{x(1-x)}{2} \frac{\partial^2 p}{\partial x^2}$$

(see [7, (1.10)] for the explicit formula for $p(t;x,y)$). From this fact and (12), it is not hard to see that

$$L(t)(f,x) = f(0) + \int_0^\infty [f(y) - f(0)]\, q(t;x,y)\, dy,$$

where

$$q(t; x, y) := \frac{1 + x}{(1 + y)^3} p\left(t; \frac{x}{1 + x}, \frac{y}{1 + y}\right).$$

Remark 2. The results for the Bernstein operator were extended by Karlin and Ziegler [7] to a more general context of positive linear operators. Although the Baskakov operator is not specifically mentioned in [7], it satisfies analogous properties to the Szász operator (see [7, Sect. 5]). Therefore, as we have shown for $B_n^{k(n)}$ and $L_n^{k(n)}$, the properties concerning the limit behavior of $H_n^{k(n)}$ can be transferred to $M_n^{k(n)}$ via Theorem 2 above. We shall not enter into the details.

5. CONCLUDING REMARKS

Theorems 1 and 2 may have other applications. As an example, we give a simple proof of the following known results on convexity concerning L_n (cf. [3, Theorems 2.1 and 2.2; 9; 10]).

THEOREM 6. *Let $f \in \mathbb{R}^{[0,\infty)}$ be a nonincreasing convex function. Then, for $n = 0, 1, 2, \ldots,$ we have*:

(a) *$L_n f$ is convex.*

(b) *$L_n f \geq L_{n+1} f$.*

To prove Theorem 6, we shall need the following elementary result.

LEMMA 6. *The operators S, T, and T_* preserve convexity. Also, if $f \in \mathbb{R}^{[0,\infty)}$ is nonincreasing and convex, then $S_* f$ is convex.*

Proof of Theorem 6. As it is well known, if $h \in \mathbb{R}^{[0,1]}$ is convex then, for $n = 1, 2, \ldots, B_n h$ is convex and, moveover, $B_n h \geq B_{n+1} h$. Thus, the conclusion in (a) follows from Theorem 1 and Lemma 6, while part (b) is a consequence of Theorem 1, Lemma 6, and the positivity of T_*. The proof is complete. ∎

Also, the operators H_n and M_n preserve convexity and have the property of monotonic convergence under convexity (cf. [2, 10]). In view of Theorem 2 and Lemma 6, it becomes apparent that M_n satisfies these properties if and only if H_n does.

On the other hand, we can obtain characterizations of convexity for L_n and M_n similar to those established by Karlin and Ziegler in [7, Sect. 7]. This can be done by using the results in Section 4 or, alternatively, by using the corresponding known results for B_n and H_n, via Theorems 1 and 2 and Lemmas 1 and 6. Details are omitted.

REFERENCES

1. G. Bleimann, P. L. Butzer, and L. Hahn, A Bernstein-type operator approximating continuous functions on the semi-axis, *Indag. Math.* **42** (1980), 255–262.
2. E. W. Cheney and A. Sharma, Bernstein power series, *Canad. J. Math.* **16** (1964), 241–252.
3. B. Della Vecchia, Some properties of a rational operator of Bernstein-type, *in* "Progress in Approximation Theory" (P. Nevai and A. Pinkus, Eds.), pp. 177–185, Academic Press, New York, 1991.
4. F. Felbecker, Linearkombinationen von iterierten Bernsteinoperatoren, *Manuscripta Math.* **29** (1979), 229–248.
5. W. Gawronsky and U. Stadtmüller, Linear combinations of iterated generalized Bernstein functions with an application to density estimation, *Acta Sci. Math. (Szeged)* **47** (1984), 205–221.
6. N. L. Johnson and S. Kotz, "Discrete Distributions," Houghton Mifflin, Boston, 1969.
7. S. Karlin and Z. Ziegler, Iteration of positive approximation operators, *J. Approx. Theory* **3** (1970), 310–339.
8. R. P. Kelisky and T. J. Rivlin, Iterates of Bernstein polynomials, *Pacific J. Math.* **21** (1967), 511–520.
9. R. A. Khan, Some properties of a Bernstein-type operator of Bleimann, Butzer and Hahn, *in* "Progress in Approximation Theory" (P. Nevai and A. Pinkus, Eds.), pp. 497–504, Academic Press, New York, 1991.
10. R. A. Khan, Reverse martingales and approximation operators, *J. Approx. Theory* **80** (1995), 367–377.
11. G. G. Lorentz, "Bernstein Polynomials," 2nd ed., Chelsea, New York, 1986.
12. G. Mastroianni and M. R. Occorsio, Una generalizzazione dell'operatore di Bernstein, *Rend. Accad. Sci. Fis. Mat. Napoli (4)* **44** (1977), 151–169.
13. C. A. Micchelli, The saturation class of iterates of the Bernstein polynomials, *J. Approx. Theory* **8** (1973), 1–18.
14. V. Totik, Uniform approximation by Bernstein-type operators, *Indag. Math.* **46** (1984), 87–93.